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A Markov Process Associated with a Boltzmann Equation Without Cutoff and for Non-Maxwell Molecules

Nicolas Fournier[1] **and Sylvie Méléard**[2]

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Tanaka,[18] showed a way to relate the measure solution $\{P_t\}_t$ of a spatially homogeneous Boltzmann equation of Maxwellian molecules without angular cutoff to a Poisson-driven stochastic differential equation: $\{P_t\}$ is the flow of time marginals of the solution of this stochastic equation. In the present paper, we extend this probabilistic interpretation to much more general spatially homogeneous Boltzmann equations. Then we derive from this interpretation a numerical method for the concerned Boltzmann equations, by using easily simulable interacting particle systems.

KEY WORDS: Boltzmann equations without cutoff; nonlinear stochastic differential equations; jump measures; interacting particle systems.

1. INTRODUCTION

The spatially homogeneous Boltzmann equation deals with the distribution of the velocities $P_t(dv)$ at the instant t, in a gas. In the case of Maxwell molecules, Tanaka,[18] has built a process V_t, which can be seen as the velocity of the "mean particle," of which the law is given by $P_t(dv)$. This representation of this particular Boltzmann equation has proved very useful. Firstly, it did allow to extend the works of Graham, Méléard,[9, 14] which were concerning numerical methods for Boltzmann equations with cutoff, to the case of Boltzmann equations without cutoff, see Desvillettes, Graham, Méléard,[3] and Fournier, Méléard.[6] Secondly, the use of recent tools of stochastic analysis did allow to prove, via Tanaka's representation,

[1] Institut Elie Cartan, Campus Scientifique, BP 239, 54506 Vandoeuvre-lès-Nancy Cedex, France; e-mail: fournier@iecn.u-nancy.fr

[2] Université Paris 10, MODALX, 200 av. de la République, 92000 Nanterre et ASCI (UPR 9029), Bat. 506 Université Paris Sud, 91405 Orsay cedex, France; e-mail: sylm@ccr.jussieu.fr

the existence of very smooth and positive solutions to the Boltzmann equation, see the works of Graham, Méléard,[10] Fournier.[5]

Our aim in this paper is to extend the probabilistic interpretation of Tanaka to the case of non-Maxwell molecules. We thus consider a quite general spatially homogeneous 2-dimensional Boltzmann equation without angular cutoff. Then we state a nonlinear stochastic differential equation of Poisson type, related to our Boltzmann equation.

We prove, by using the usual tools of convergence in law on the set of càdlàg functions, the existence of a solution to this stochastic equation. As a corollary, we obtain a new result of existence of weak solutions of Boltzmann equations, in particular with very general initial data.

Using the above developed tools, we then build Monte Carlo approximations. We prove a convergence result of the empirical law associated with an interacting particle system to the solution of our Boltzmann equation.

We finally discuss about numerical results.

There is a very tiny literature about existence, uniqueness or regularity of weak solutions for a Boltzmann equation without cutoff, for a non-Maxwell gas, even in the spatially homogeneous case. Existence results can be found in Goudon,[8] and Villani,[19] and a regularity result has been obtained by Alexandre *et al.*[1] Let us also quote Proutière.[16]

Uniqueness is an open problem for the equation we consider. Since all the convergence results we prove are obtained by compactness methods, we obtain only existence of converging subsequences.

We study here the $2D$ case for technical (but rather serious) reasons. From our probabilistic point of view, the study of the $3D$ Boltzmann equations is much more difficult, because the collisions make appear discontinuous coefficients in the velocity variable. See ref. 6 for more details. Let us finally remark that although the approach of Tanaka has allowed many nice results in the spatially homogeneous case, the extension of this approach to the full Boltzmann equation seems impossible. We however hope to be able to deal one day with the case of the "mollified" inhomogeneous Boltzmann equation, in which the interactions are delocalized.

The paper is organized as follows: in the next section, we recall the Bolzmann equation. In Section 3, we give our pathwise interpretation, and solve the nonlinear Poisson driven stochastic differential equation. In Section 4, we study particle systems. We describe the (very simple) simulation algorithm in Section 5. Numerical results are given in Section 6.

Notations.

— $\mathbb{D}_T$ will denote the Skorohod space $\mathbb{D}([0,T],\mathbb{R}^2)$ of càdlàg functions from $[0,T]$ into $\mathbb{R}^2$. The space $\mathbb{D}_T$ endowed with the Skorohod topology is a Polish space.

— $\mathscr{P}(\mathbb{R}^2)$ is the set of probability measures on $\mathbb{R}^2$ and $\mathscr{P}_2(\mathbb{R}^2)$ the subset of probability measures with a second order moment. Similarly, $\mathscr{P}(\mathbb{D}_T)$ will denote the space of probability measures on $\mathbb{D}_T$ and $\mathscr{P}_2(\mathbb{D}_T)$ will be the subset of probability measures with a second order moment: $q \in \mathscr{P}_2(\mathbb{D}_T)$ if $\int_{x \in \mathbb{D}_T} \sup_{t \in [0,T]} |x(t)|^2 \, q(dx) < \infty$.

— K will denote a real positive constant of which the value may change from line to line.

2. THE EQUATION

The Boltzmann equation we consider describes the evolution of the density $f(t, v)$ of particles with velocity $v \in \mathbb{R}^2$ at time t in a rarefied homogeneous 2-dimensional gas:

$$\frac{\partial f}{\partial t} = Q(f, f), \tag{2.1}$$

where Q is a quadratic collision kernel preserving momentum and kinetic energy, of the form

$$Q(f, f)(t, v) = \int_{v_* \in \mathbb{R}^2} \int_{\theta = -\pi}^{\pi} (f(t, v') \, f(t, v'_*) - f(t, v) \, f(t, v_*)) \times B(|v - v_*|, \theta) \, d\theta \, dv_*$$

with

$$v' = v + A(\theta)(v - v_*); \qquad v'_* = v_* - A(\theta)(v - v_*) \tag{2.2}$$

and

$$A(\theta) = \frac{1}{2} \begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix}$$

Remark 2.1. For each $\theta, \varphi \in [-\pi, \pi] \backslash \{0\}$,

$$|A(\theta)| \leqslant K \, |\theta| \qquad \text{and} \qquad |A(\theta) - A(\varphi)| \leqslant K \, |\theta - \varphi|. \tag{2.3}$$

The cross-section B is a positive function. In the $3D$ situation, if the molecules in the gas interact according to an inverse power law in $1/r^s$ with $s \geqslant 2$, then $B(z, \theta) = z^{\frac{s-5}{s-1}} d(|\theta|)$ where $d \in L^\infty_{loc}(]0, \pi])$ and $d(\theta) \sim K(s) \, \theta^{-\frac{s+1}{s-1}}$ when θ goes to zero, for some $K(s) > 0$. Physically, this explosion comes from the accumulation of grazing collisions.

In this general (spatially homogeneous) setting, the Boltzmann equation is very difficult to study. A large literature deals with the non physical equation with angular cutoff, namely under the assumption $\int_0^\pi B(z,\theta)\,d\theta < \infty$. More recently, the case of Maxwell molecules, for which the cross section $B(z,\theta)=\beta(\theta)$ only depends on θ, has been much studied without the cutoff assumption. In the Maxwell context, Tanaka,[18] was considering the case where $\int_0^\pi \theta\beta(\theta)\,d\theta<\infty$, and Desvillettes,[2] Desvillettes, Graham, Méléard,[3] and Fournier,[5] have worked under the general physical assumption $\int_0^\pi \theta^2\beta(\theta)\,d\theta<+\infty$.

We will consider here cross sections of the following type, by analogy of what happens in the 3D situation.

Hypothesis (R). The cross section can be written as

$$B(z,\theta)=\psi(z)\,\beta(\theta), \qquad \text{with} \tag{2.4}$$

(1) β even from $[-\pi,\pi]/\{0\}\to\mathbb{R}+$ and such that $\int_{-\pi}^{\pi}|\theta|\,\beta(\theta)\,d\theta < \infty$;

(2) ψ positive function and locally Lipschitz continuous and $\psi(z)\leqslant M$, where $M\in\mathbb{R}+$.

Notice that we are still far from the physical situations (that is, 3D cross sections whose kinetic part explodes at 0 or at ∞).

We will see that our approach does not allow us to consider functions β with just a second order moment. In a work in preparation we consider another case for which β just integrates θ^2, but with more restrictive assumption on z for the cross-section. Here, hypotheses on ψ are not very stringent, except its boundedness. In particular, the strict positivity of ψ outside 0 is not required.

Equation (2.1) has to be understood in a weak sense, i.e., f is a solution of the equation if for each test function $\phi\in C_b^1(\mathbb{R}^2)$ (the set of C^1 functions on $\mathbb{R}^2$ whose derivative is bounded),

$$\frac{\partial}{\partial t}\langle f,\phi\rangle=\langle Q(f,f),\phi\rangle$$

where $\langle\cdot,\cdot\rangle$ denotes the duality bracket between L^1 and L^∞ functions. A standard integration by parts shows that f satisfies for each $\phi\in C_b^1(\mathbb{R}^2)$

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^2} f(t,v)\,\phi(v)\,dv=\int_{\mathbb{R}^2\times\mathbb{R}^2}\int_{-\pi}^{\pi}(\phi(v')-\phi(v))\\ \times\psi(|v-v_*|)\,\beta(\theta)\,d\theta\,f(t,v)\,dv\,f(t,v_*)\,dv_* \tag{2.5}$$

We have conservation of mass in (2.5), which leads to the following definition of solutions of (2.1).

Definition 2.2. Consider P_0 a probability measure on $\mathbb{R}^2$. We say that a probability measure flow $(P_t)_t$ is a measure-solution of the Boltzmann equation (2.1) with initial data P_0 if for each $\phi \in C_b^1(\mathbb{R}^2)$

$$\langle \phi, P_t \rangle = \langle \phi, P_0 \rangle + \int_0^t \langle K_\beta^\phi(v, v^*), P_s(dv)\, P_s(dv^*) \rangle\, ds, \tag{2.6}$$

where K_β^ϕ is defined by

$$K_\beta^\phi(v, v^*) = \int_{-\pi}^{\pi} (\phi(v + A(\theta)(v - v_*)) - \phi(v))\, \psi(|v - v_*|)\, \beta(\theta)\, d\theta. \tag{2.7}$$

The probabilitistic approach consists in considering (2.6) as the evolution equation of the flow of marginals of a Markov process. The law of this process will be solution of the following nonlinear martingale problem.

Definition 2.3. Let B be a cross section satisfying Hypothesis (R) and let P_0 belong to $\mathscr{P}_2(\mathbb{R}^2)$. We say that $P \in \mathscr{P}_2(\mathbb{D}_T)$ solves the nonlinear martingale problem (MP) starting at P_0 if for X the canonical process under P, the law of X_0 is P_0 and for any $\phi \in C_b^1(\mathbb{R}^2)$,

$$\phi(X_t) - \phi(X_0) - \int_0^t \langle K_\beta^\phi(X_s, v_*), P_s(dv_*) \rangle\, ds \tag{2.8}$$

is a square-integrable martingale. Here, the nonlinearity appears through P_s which denotes the marginal of P at time s.

Remark 2.4. Taking expectations in (2.8), we remark that if P is a solution of (MP), then its marginal flow $(P_t)_t$ is a measure-solution of the Boltzmann equation, in the sense of Definition 2.2.

Our first aim is to prove the existence of a solution to the martingale problem (2.8) and then to obtain the existence of a measure-solution to the Boltzmann equation. Our method gives no hope to obtain a uniqueness result. We will also introduce a specific nonlinear stochastic differential equation giving a pathwise version of the probabilistic interpretation. We will study the existence of solutions of this equation, first in a weak sense under Hypothesis (R) and next in a strong sense under more stringent assumptions.

Then we will use this pathwise probabilistic interpretation of the solutions to show that these solutions can be obtained as limits of the laws of stochastic interacting particle systems, and we will deduce a very simple Monte-Carlo algorithm of simulation for the solutions.

3. A PATHWISE APPROACH

Let us now consider two probability spaces: the first one is the abstract space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \in [0,T]}, P)$ and the second one is $([0,1], \mathscr{B}([0,1]), d\alpha)$. In order to avoid any confusion, the processes on $([0,1], \mathscr{B}([0,1]), d\alpha)$ will be called α-processes, the expectation under $d\alpha$ will be denoted by E_α, and the laws $\mathscr{L}_\alpha$.

Notation 3.1. We will denote by L^2_T the space of $\mathbb{D}_T$-valued processes Y such that $E(\sup_{t \in [0,T]} |Y_t|^2) < +\infty$ and by $L^2_T - \alpha$ the space of α-processes W such that $E_\alpha(\sup_{t \in [0,T]} |W_t|^2) < +\infty$.

Definition 3.2. We will say that (V, W, N, V_0) is a solution of (SDE) if

(i) (V_t) is an adapted L^2_T-process,

(ii) (W_t) is a $L^2_T - \alpha$-process,

(iii) $N(\omega, dt, d\alpha, dz, d\theta)$ is a $\{\mathscr{F}_t\}$-Poisson measure on $[0,T] \times [0,1] \times [0,M] \times [-\pi, \pi]$ with intensity $m(dt, d\alpha, dz, d\theta) = dt\, d\alpha\, dz\, \beta(\theta)\, d\theta$,

(iv) V_0 is a square integrable variable independent of N,

(v) $\mathscr{L}(V) = \mathscr{L}_\alpha(W)$,

(vi)

$$V_t = V_0 + \int_0^t \int_0^1 \int_0^M \int_{-\pi}^{\pi} A(\theta)(V_{s-} - W_{s-}(\alpha))\, \mathbf{1}_{\{z \leqslant \psi(|V_{s-} - W_{s-}(\alpha)|)\}} N(ds, d\alpha, dz, d\theta) \tag{3.1}$$

This definition can be understood through the following remark.

Remark 3.3. If (V, W, N, V_0) is a solution of (SDE), one easily proves, by using the Itô formula, that $\mathscr{L}(V) = \mathscr{L}_\alpha(W)$ is a solution of (MP) with initial law $Q_0 = \mathscr{L}(V_0)$, and thus $\{\mathscr{L}(V_s)\}_{s \in [0,T]}$ is a measure-solution of the Boltzmann equation (2.6) with initial data Q_0.

We are now able to state some existence results, which are the main of this section.

Theorem 3.4. Assume that Q_0 is a probability measure on $\mathbb{R}^2$ admitting a moment of order 2, and that $B(x, \theta) = \psi(x)\, \beta(\theta)$ is a cross-section satisfying Hypothesis (R). Then (1) The nonlinear martingale problem (MP) with initial data Q_0 admits a solution $Q \in \mathscr{P}_2(\mathbb{D}_T)$. (2) Let Q be any solution of (MP). Let W be any α-process such that $\mathscr{L}_\alpha(W) = Q$. On an enlarged probability space from the canonical space $(\mathbb{D}_T, \mathscr{D}_T, Q)$ there exist a Poisson measure N with intensity m and an independent square integrable variable V_0 with law Q_0 such that (X, W, N, V_0) is solution of (SDE), where X is the canonical process. (That means that there exists a weak solution to (SDE)). (3) If one assumes moreover that:

Hypothesis (CL). The function ψ is locally Lipschitz continuous, with a Lipschitz constant linearly increasing, i.e.,

$$|\psi(x) - \psi(y)| \leqslant K\, |x-y|\, (1 + |x| + |y|),$$

then there exists a strong solution to the nonlinear stochastic differential equation (SDE): for each probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \in [0,T]}, P)$, for each Poisson measure N with intensity m and each square integrable variable V_0 independent of N, there exist V, W such that (V, W, N, V_0) is solution of (SDE).

Remark 3.5. There is no assumption on Q_0, except the existence of a second order moment. This allows in particular to consider degenerate initial data, as Dirac measures. The point (1) in Theorem 3.4 exhibits in particular a measure-solution to the Boltzmann equation (2.1) for every initial data $Q_0 \in \mathscr{P}_2(\mathbb{R}^2)$. The point (2) in Theorem 3.4 gives a stochastic "pathwise" interpretation of the solution which might be helpful to study this measure-solution. The point (3) in Theorem 3.4 might be useful in situations where a "strong" existence is needed, as for examples coupling techniques.

Before proving these results, let us mention that the obtained existence result is not strong, at least from the Boltzmann equation point of view. Indeed, Villani has proved in ref. 19 the existence of a function-solution f to the $3D$ Boltzmann equation (and even in any dimension $d \geqslant 2$), under the conditions that: there exists $\gamma \in [-4, 1[$, such that $\psi(z) = |z|^\gamma$, there exists $\nu > 0$ such that $\beta(\theta) \sim \theta^{-(1+\nu)}$ when θ goes to 0, the quantity

$\int \theta^2 \beta(\theta)\, d\theta$ is finite, and the initial distribution Q_0 has a density f_0 satisfying a condition of finite energy and entropy.

A Sobolev regularity result of type $f(t, v) \in H^{\nu/2}_{loc}(\mathbb{R}^3)$ for all $t > 0$ is also proved in Alexandre *et al.*[1]

Our approach allows to consider more general initial conditions, but our assumptions on ψ are much more stringent. The main interest of our approach is to give a probabilistic interpretation to the equation, which allows in particular to obtain a numerical approximation scheme. Furthermore, a pathwise study of the process $\{V_t\}_{t \geqslant 0}$ would allow to understand the "mean" behaviour of the particles in the gas. Finally, existence, smoothness, and positivity of function-solutions might be obtained by applying Malliavin Calculus techniques to the stochastic process $\{V_t\}_{t \geqslant 0}$. We are far, for the moment, from proving such results, because of the indicator function that appears in (3.1). We however work in this direction, and at least existence results should be obtained.

Let us now come back to our problem. We begin with some notation. We denote by φ the function from $\mathbb{R}^2 \times \mathbb{R}^2 \times [0, M]$ into $\mathbb{R}^2$ defined as

$$\varphi(v, w, z) = (v - w)\, \mathbf{1}_{\{z \leqslant \psi(|v-w|)\}}. \tag{3.2}$$

Notice that the collision kernel K^{ϕ}_{β} defined in (2.7) can be written as

$$K^{\phi}_{\beta}(v, v^*) = \int_{-\pi}^{\pi} \int_0^M (\phi(v + A(\theta)\, \varphi(v, v_*, z)) - \phi(v))\, dz\, \beta(\theta)\, d\theta \tag{3.3}$$

and that (3.1) can be written as

$$V_t = V_0 + \int_0^t \int_0^1 \int_0^M \int_{-\pi}^{\pi} A(\theta)\, \varphi(V_{s-}, W_{s-}(\alpha), z)\, N(ds, d\alpha, dz, d\theta) \tag{3.4}$$

We now give the proof of Theorem 3.4 which is obtained in many steps.

We first introduce, for $n \in \mathbb{N}^*$, the functions φ_n from $\mathbb{R}^2 \times \mathbb{R}^2 \times [0, M]$ into $\mathbb{R}^2$ defined by

$$\varphi_n(v, w, z) = \varphi(v \wedge n \vee (-n), w \wedge n \vee (-n), z) \tag{3.5}$$

where $v \wedge n$ (resp. $v \vee (-n)$), denotes the vector $(v_1 \wedge n, v_2 \wedge n)$ (resp. $(v_1 \vee (-n), v_2 \vee (-n))$, if $v = (v_1, v_2)$). The functions φ and φ_n satisfy the following properties.

Lemma 3.6. Under (R),

$$\int_0^M |\varphi(v, w, z)|\, dz \leqslant M\, |v-w|\, ; \qquad \int_0^M |\varphi_n(v, w, z)|\, dz \leqslant M\, |v-w|$$

$$\int_0^M |\varphi(v, w, z) - \varphi(v', w', z)|\, dz \leqslant M(|v-v'| + |w-w'|)$$
$$+ |v-w|\, (\psi(|v-w|) - \psi(|v'-w'|))$$

$$\int_0^M |\varphi_n(v, w, z) - \varphi_n(v', w', z)|\, dz \leqslant K_n(|v-v'| + |w-w'|)$$

with K_n a constant depending on n.

The proof of this lemma is easy and left to the reader.

Similarly to Definition 3.2, we consider the equation $(SDE)_n$ defined by replacing φ by φ_n:

$$V_t^n = V_0 + \int_0^t \int_0^1 \int_0^M \int_{-\pi}^{\pi} A(\theta)\, \varphi_n(V_{s-}^n - W_{s-}^n(\alpha))\, N(ds, d\alpha, dz, d\theta)$$
$$\text{with} \quad \mathscr{L}(V^n) = \mathscr{L}_\alpha(W^n)$$

In the same way, we will denote by $K_\beta^{n,\phi}$ the kernel defined as (3.3) with φ replaced by φ_n.

Proposition 3.7. Assume (R) and consider $Q_0 \in \mathscr{P}_2(\mathbb{R}^2)$. For each pair (V_0, N), V_0 being with law Q_0 and N a Poisson measure with intensity m, the equation $(SDE)_n$ admits a solution (V^n, W^n, N, V_0) and

$$\sup_{n \geqslant 1} E\Big(\sup_{t \in [0,T]} |V_t^n|^2\Big) < +\infty. \tag{3.6}$$

Moreover, $Q^n = \mathscr{L}(V^n) = \mathscr{L}_\alpha(W^n)$ is the unique solution of the nonlinear martingale problem $(\mathrm{MP})_n$ wich is similar to (2.8) with $K_\beta^\phi(v, v_*)$ replaced by $K_\beta^{n,\phi}(v, v_*)$.

Proof. We fix $n \geqslant 1$, V_0 with law Q_0 and independent of a Poisson measure N with intensity measure m. Following Tanaka,[18] Desvillettes–Graham–Meleard[3] or Fournier,[5] we construct a specific Picard iteration which allows us to obtain the existence of a pair (V^n, W^n) such that (V^n, W^n, N) is a solution of $(SDE)_n$. We first consider the process X^0

identically equal to V_0, then consider Y^0 defined on $[0, 1]$ such that $\mathscr{L}_\alpha(Y^0) = \mathscr{L}(X^0)$. By induction, assuming that $X^0, X^1, ..., X^k$ and $Y^0, Y^1, ..., Y^k$ are constructed, one defines X^{k+1} by

$$X_t^{k+1} = V_0 + \int_0^t \int_0^1 \int_0^M \int_{-\pi}^{\pi} A(\theta)\, \varphi_n(X_{s-}^k, Y_{s-}^k(\alpha), z)\, N(ds, d\alpha, dz, d\theta)$$

and one considers on $[0, 1]$ a process Y^{k+1} such that

$$\mathscr{L}_\alpha(Y^0, Y^1, ..., Y^{k+1}) = \mathscr{L}(X^0, X^1, ..., X^{k+1})$$

and so on. One proves easily thanks to Lemma 3.6 that for each fixed n,

$$\begin{aligned} E\Big(\sup_{t \in [0,T]} |X_t^{k+1} - X_t^k|\Big) &\leqslant \int_0^t \int_0^1 \int_0^M \int_{-\pi}^{\pi} |A(\theta)|\, E(|\varphi_n(X_{s-}^k, Y_{s-}^k(\alpha), z) \\ &\qquad - \varphi_n(X_{s-}^{k-1}, Y_{s-}^{k-1}(\alpha), z)|)\, ds\, d\alpha\, dz\, \beta(\theta)\, d\theta \\ &\leqslant K_n \int_0^t E\Big(\sup_{u \leqslant s} |X_u^k - X_u^{k-1}|\Big)\, ds. \end{aligned} \tag{3.7}$$

We deduce easily that there exist an adapted process X with $E(\sup_{t \in [0,T]} |X_t|) < \infty$ and a α-process Y with $\mathscr{L}_\alpha(Y) = \mathscr{L}(X)$ and $E(\sup_{t \in [0,T]} |X_t^k - X_t|) = E_\alpha(\sup_{t \in [0,T]} |Y_t^k - Y_t|)$ tends to zero as k tends to infinity. Then (X, Y, N, V_0) is solution of $(SDE)_n$. Moreover, since $|A(\theta)| \in L^1 \cap L^\infty(\beta(\theta)\, d\theta)$, and thanks to Lemma 3.6, one proves that since $\mathscr{L}(V_0) = Q_0$ admits a second order moment,

$$E\Big(\sup_{t \leqslant T} |X_t|^2\Big) < +\infty.$$

Now, let us rename $X = V^n$ and denote by Q^n the law of V^n.

The proof of the uniqueness in law of a solution of $(SDE)_n$ is obtained by a coupling argument, exactly as in ref. 3, Theorems 3.6-2 and 3.7. One first proves that the law $\mathscr{L}(V^n) = \mathscr{L}_\alpha(W^n)$ of the solution of the nonlinear stochastic differential equation $(SDE)_n$ obtained by the Picard iteration does not depend on the possible choices for Ω, V_0, N and next, one shows that if $(U, W, \hat{N}, V_0)$ is a solution of $(SDE)_n$, then $\mathscr{L}(U) = \mathscr{L}(V^n) = Q^n$, where $(V^n, Y, \hat{N}, V_0)$ is the Picard iteration constructed on the probability space associated with V_0 and $\hat{N}$.

Now, to obtain the uniqueness of the solution of $(\mathrm{MP})_n$, consider another solution R of $(\mathrm{MP})_n$. One shows, by using a comparison between the Itô formula and the martingale problem, that the canonical process

X is under R a pure jump process and that its Lévy measure is the image measure of the measure $m(ds, d\alpha, dz, d\theta) = ds\, d\alpha\, dz\, \beta(\theta)\, d\theta$ by the mapping $(\theta, \alpha, z, s) \mapsto A(\theta)\, \varphi(X_{s-}, W_{s-}(\alpha), z)$, where the process $W_s(\alpha)$ is a process chosen on the probability space $[0, 1]$ with law R. Then, by using the representation theorem proved in Grigelionis[11] and El Karoui–Lepeltier[4] (see also ref. 17), we know that there exist on an enlarged probability space a square integrable variable V_0 and an independent point Poisson measure N with intensity m such that (X, W, N, V_0) is a solution of $(SDE)_n$. Then by the uniqueness proved above, R is equal to Q^n and the martingale problem $(\mathrm{MP})_n$ has a unique solution.

It remains to prove (3.6). Since $\int_0^M |\varphi_n(v, w, z)|\, dz \leqslant K(|v| + |w|)$, with K independent of n, since $\mathscr{L}_\alpha(W^n) = \mathscr{L}(V^n)$, we show that

$$E(\sup_{s \leqslant t} |V_s^n|^2) \leqslant E(|V_0|^2) + K \int_0^t E(\sup_{u \leqslant s} |V_u^n|^2)\, ds$$

where K does not depend on n, and Gronwall's lemma allows to conclude. ∎

Proposition 3.8. Under Hypothesis (R), still assuming that $Q_0 \in \mathscr{P}_2(\mathbb{R}^2)$, the sequence of probability measures $(Q^n)_n$ on $\mathbb{D}_T$ obtained in Proposition 3.7 is tight for the weak convergence on $\mathscr{P}_2(\mathbb{D}_T)$, and any limit point Q of $(Q^n)_n$ is solution of the nonlinear martingale problem (MP).

Proof. (1) We prove that the sequence Q^n is tight for the weak convergence on $\mathscr{P}_2(\mathbb{D}_T)$. Thanks to (3.6), we just need to verify the Aldous criterion (see, e.g., Jacod–Shiryaev,[12] p. 320). We have, for stopping times τ and τ' with $0 \leqslant \tau \leqslant \tau' \leqslant \tau + \delta$,

$$\begin{aligned}
E(|V_{\tau'}^n - V_\tau^n|) &\leqslant E\left(\int_\tau^{\tau'} \int_0^1 \int_0^M \int_{-\pi}^{\pi} |A(\theta)|\, |\varphi_n(V_{s-}^n, W_{s-}^n(\alpha), z)|\, \beta(\theta)\, d\theta\, dz\, d\alpha\, ds \right) \\
&\leqslant KE\left(\int_\tau^{\tau'} \int_0^1 (|V_{s-}^n| + |W_{s-}^n(\alpha)|)\, d\alpha\, ds \right) \\
&\qquad \left(\text{since } \int |A(\theta)|\, \beta(\theta)\, d\theta < \infty \right) \\
&\leqslant KE((\tau' - \tau) \sup_{t \leqslant T} |V_t^n|) + KE(\tau' - \tau)\, E_\alpha(\sup_{t \leqslant T} |W_t^n|) \leqslant K\delta
\end{aligned}$$

by (3.6), where K is independent of n. Then we deduce that for each $\eta > 0$,

$$\sup_n \sup_{\{\tau, \tau';\, 0 \leqslant \tau \leqslant \tau' \leqslant \tau + \delta\}} P(|V_{\tau'}^n - V_\tau^n| \geqslant \eta)$$

tends to 0 as δ tends to 0, and the Aldous criterion is satisfied. Hence the sequence (Q^n) is tight.

(2) Let us now identify each limit point of (Q^n). Let Q be a limit value of this sequence. We consider the canonical process $(X_t)_t$ on $\mathbb{D}_T$ and for $\phi \in C_b^1(\mathbb{R}^2)$, $t > 0$, we set

$$H_t^\phi = \phi(X_t) - \phi(X_0) - \int_0^t \int_0^M \int_{-\pi}^{\pi} \int_{w \in \mathbb{R}^2} (\phi(X_u + A(\theta)\, \varphi(X_u, w, z)) - \phi(X_u))\, Q_u(dw)\, \beta(\theta)\, d\theta\, dz\, du$$

and $H_t^{n,\phi}$ denotes a similar quantity with φ_n instead of φ and Q^n instead of Q. The probability measure Q will be a solution of the nonlinear martingale problem (MP) with initial law Q_0 if it satisfies for each $0 \leqslant s_1 < \cdots < s_k < s < t \leqslant T$, each $g_1, \ldots, g_k \in C_b(\mathbb{R}^2)$,

$$\langle (H_t^\phi - H_s^\phi)\, g_1(X_{s_1}) \cdots g_k(X_{s_k}), Q \rangle = 0. \tag{3.8}$$

Since Q^n is solution of $(\mathrm{MP})_n$, we already know that

$$\langle (H_t^{n,\phi} - H_s^{n,\phi})\, g_1(X_{s_1}) \cdots g_k(X_{s_k}), Q^n \rangle = 0.$$

Since the sequence (Q^n) satisfies the Aldous criterion, the law Q is the law of a quasi-càg process (cf. ref. 12, p. 321). Then the mapping $F: x \mapsto (\phi(x_t) - \phi(x_s))\, g_1(x_{s_1}) \cdots g_k(x_{s_k})$ is Q-a.e. continuous and bounded from $\mathbb{D}_T$ to $\mathbb{R}$. Thus $\langle F, Q^n \rangle$ tends to $\langle F, Q \rangle$ as n tends to infinity. Next, let us prove that α_n defined by

$$\left\langle \left(\int_s^t \int_0^M \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} (\phi(X_u + A(\theta)\, \varphi(X_u, w, z)) - \phi(X_u + A(\theta)\, \varphi_n(X_u, w, z))) \times dz\, Q_u^n(dw)\, \beta(\theta)\, d\theta\, du \right) g_1(X_{s_1}) \cdots g_k(X_{s_k}), Q^n \right\rangle$$

tends to 0 as n tends to infinity. We have:

$$\begin{aligned} &|\phi(X_u + A(\theta)\, \varphi(X_u, w, z)) - \phi(X_u + A(\theta)\, \varphi_n(X_u, w, z))| \\ &\quad \leqslant \|\nabla \phi\|_\infty\, |A(\theta)| \int_0^M |\varphi(X_u, w, z) - \varphi_n(X_u, w, z)|\, dz \\ &\quad \leqslant K\, |\theta|\, (|w| + |X_u|)(\mathbf{1}_{\{|X_u| \geqslant n\}} + \mathbf{1}_{\{|w| \geqslant n\}}). \end{aligned}$$

Then

$$|\alpha_n| \leqslant K\Pi_{i=1,\dots,k} \|g_i\|_\infty \left\langle \int_s^t \int_{\mathbb{R}^2} (|w|+|X_u|)(\mathbf{1}_{\{|X_u|\geqslant n\}} + \mathbf{1}_{\{|w|\geqslant n\}})\, Q_u^n(dw)\, du,\, Q^n \right\rangle$$

$$\leqslant K \int_{x\in\mathbb{D}_T} \int_{y\in\mathbb{D}_T} (\sup_{t\leqslant T} |x(t)| + \sup_{t\leqslant T} |y(t)|)(\mathbf{1}_{\{\sup_{t\leqslant T}|x_t|\geqslant n\}} + \mathbf{1}_{\{\sup_{t\leqslant T}|y_t|\geqslant n\}})$$

$$\times Q^n(dx)\, Q^n(dy)$$

$$\leqslant K\left(\left(\int_{x\in\mathbb{D}_T} (\sup_{t\leqslant T} |x(t)|)\, Q^n(dx)\right)\times\left(\int_{x\in\mathbb{D}_T} (\mathbf{1}_{\{\sup_{t\leqslant T}|x_t|\geqslant n\}})\, Q^n(dx)\right)\right.$$

$$\left. + \int_{x\in\mathbb{D}_T} (\sup_{t\leqslant T} |x(t)|)(\mathbf{1}_{\{\sup_{t\leqslant T}|x_t|\geqslant n\}})\, Q^n(dx)\right).$$

By (3.6), we show that $\int_{x\in\mathbb{D}_T}(\sup_{t\leqslant T}|x(t)|)\, Q^n(dx)$ is bounded uniformly in n, that $\int_{x\in\mathbb{D}_T}(\mathbf{1}_{\{\sup_{t\leqslant T}|x_t|\geqslant n\}})\, Q^n(dx)$ tends to 0 as n tends to infinity, and by Cauchy–Schwarz inequality that $\int_{x\in\mathbb{D}_T}(\sup_{t\leqslant T}|x(t)|)(\mathbf{1}_{\{\sup_{t\leqslant T}|x_t|\geqslant n\}})\, Q^n(dx)$ tends to 0 as n tends to infinity. Then α_n tends to 0 as n tends to infinity. It remains to prove that $\langle G(x, y), Q^n(dx)\otimes Q^n(dy)\rangle$ tends to $\langle G(x, y), Q(dx)\otimes Q(dy)\rangle$, where

$$G(x, y) = \left(\int_s^t \int_0^M \int_{-\pi}^{\pi} (\phi(x_u + A(\theta)\,\varphi(x_u, y_u, z)) - \phi(x_u))\, \beta(\theta)\, d\theta\, dz\, du\right)$$

$$\times g_1(x_{s_1})\cdots g_k(x_{s_k}).$$

The measure $Q^n\otimes Q^n$ converges obviously to $Q\otimes Q$. The function G is $Q\otimes Q$-a.e. continuous by a similar argument as before but not bounded. Properties in Lemma 3.6 give that

$$|G(x, y)| \leqslant K(\sup_{t\leqslant T}|x(t)| + \sup_{t\leqslant T}|y(t)|).$$

Then, for each fixed real positive number C, the sequence $\langle G\wedge C, Q^n\otimes Q^n\rangle$ converges to $\langle G\wedge C, Q\otimes Q\rangle$. We remark that

$$|G(x, y)|\, \mathbf{1}_{\{|G(x,y)|\geqslant C\}}$$

$$\leqslant K(\sup_{t\leqslant T}|x(t)| + \sup_{t\leqslant T}|y(t)|)\, \mathbf{1}_{\{\sup_{t\leqslant T}|x(t)| + \sup_{t\leqslant T}|y(t)|\geqslant C/K\}}$$

$$\leqslant K(\sup_{t\leqslant T}|x(t)| + \sup_{t\leqslant T}|y(t)|)(\mathbf{1}_{\{\sup_{t\leqslant T}|x(t)|\geqslant C/2K\}} + \mathbf{1}_{\{\sup_{t\leqslant T}|y(t)|\geqslant C/2K\}}).$$

We have already seen that

$$\sup_n \langle (\sup_{t\leqslant T} |x(t)| + \sup_{t\leqslant T} |y(t)|)(\mathbf{1}_{\{\sup_{t\leqslant T}|x(t)|\geqslant C/2K\}} + \mathbf{1}_{\{\sup_{t\leqslant T}|y(t)|\geqslant C/2K\}}), Q^n \otimes Q^n \rangle$$

tends to 0 as C tends to infinity, thanks to (3.6). Now the conclusion is obvious and the proposition is proved. ∎

Remark 3.9. Proposition 3.8 proves the first point of Theorem 3.4.

Let us now deduce the point (2) of Theorem 3.3.

Proposition 3.10. Assume (R) and $Q_0 \in \mathscr{P}_2(\mathbb{R}^2)$. Let us consider the canonical space $\mathbb{D}_T$, X the canonical process and Q the solution of (MP) obtained in Proposition 3.8. Consider a α-process W such that $\mathscr{L}_\alpha(W) = Q$, then there exist a Poisson measure N with intensity m on an enlarged probability space and an independent square integrable variable V_0 such that (X, W, N, V_0) is a solution of (SDE).

Proof. The proof is exactly similar to the end of that of Proposition 3.6. Since Q is solution of a martingale problem, the canonical process X is a semimartingale under Q. Then a comparison between the Itô formula and the martingale problem proves that X is a pure jump process and that its Lévy measure is the image measure of the measure m on $[0, T]\times[0, 1]\times[0, M]\times[-\pi, \pi]$ by the mapping $(s, \alpha, z, \theta) \mapsto A(\theta) \times \varphi(X_{s-}, W_{s-}(\alpha), z) = A(\theta)(X_{s-} - W_{s-}(\alpha))\, \mathbf{1}_{\{z \leqslant \psi(|X_{s-} - W_{s-}(\alpha)|)\}}$. Then always by the representation theorem for point measures, see ref. 4, there exist on an enlarged probability space a square integrable variable V_0 and a point Poisson measure N with intensity m such that (X, W, N, V_0) is a solution of (SDE). ∎

We are now interested in the pathwise study of the stochastic differential equation (SDE) under Hypothesis (CL). Before to study this nonlinear SDE, let us introduce the associated linearized SDE.

Definition 3.11. (1) Let be given Z a $L^2_T-\alpha$-process, V_0 a square integrable variable and N a Poisson point measure with intensity m, independent of V_0 on a fixed probability space Ω. The classical SDE

$$Y_t = V_0 + \int_0^t \int_0^1 \int_0^M \int_{-\pi}^{\pi} A(\theta)\, \varphi(Y_{s-}, Z_{s-}(\alpha), z)\, N(ds, d\alpha, dz, d\theta)$$

is denoted by $(SDE)_Z$.

(2) Let X denote the canonical process on $\mathbb{D}_T$. Let us consider $R \in \mathscr{P}_2(\mathbb{D}_T)$. Then $(\mathrm{MP})_R$ denotes the classical martingale problem: for all $\phi \in C_b^1(\mathbb{R}^2)$,

$$\phi(X_t) - \phi(X_0) - \int_0^t \langle K_\beta^\phi(X_s, .), R_s \rangle \, ds$$

is a Q-martingale.

Remark 3.12. By using the Itô formula, one proves easily that if the process Y is solution of $(SDE)_Z$, then $\mathscr{L}(Y)$ is solution of $(\mathrm{MP})_{\mathscr{L}(Z)}$.

Proposition 3.13. Let us assume Hypotheses (R) and (CL) and $Q_0 \in \mathscr{P}_2(\mathbb{R}^2)$ and let us consider a $L_T^2 - \alpha$-process Z. Then the stochastic differential equation $(SDE)_Z$ admits a unique solution Y in L_T^2. Moreover, $\mathscr{L}(Y)$ is the unique solution of $(\mathrm{MP})_{\mathscr{L}(Z)}$.

Proof. We remark that the coefficients are locally Lipschitz continuous in the Y-variable and with linear growth. Then the proof is relatively standard. Indeed,

$$\begin{aligned}\int_0^M |\varphi(v, w, z) - \varphi(v', w, z)| \, dz &\leqslant \int_0^M |(v-w) - (v'-w)| \, dz \\ &\quad + \int_0^M |v-w| \, \mathbf{1}_{\{\psi(|v-w|) \leqslant z \leqslant \psi(|v'-w|)\}} dz \\ &\leqslant K \, |v-v'| \, (1 + |v|^2 + |v'|^2 + |w|^2).\end{aligned}$$

We fix n. Let us introduce $\bar{\varphi}_n(v, w, z) = \varphi(v \wedge n \vee (-n), w, z)$. We now define the stochastic differential equation $(SDE)_Z^n$ in a similar way than $(SDE)_Z$, but with $\bar{\varphi}_n$ instead of φ. One can easily show that if Y^1 and Y^2 are two L_T^2-processes, then

$$\begin{aligned}E\bigg(\int_0^t \int_0^1 \int_0^M \int_{-\pi}^{\pi} & |A(\theta)| \, |\bar{\varphi}_n(Y_{s-}^1, Z_{s-}(\alpha), z) - \bar{\varphi}_n(Y_{s-}^2, Z_{s-}(\alpha), z)| \\ & \times N(ds, d\alpha, dz, d\theta) \bigg) \\ & \leqslant K_n E\left(\int_0^t \int_0^1 |Y_s^1 - Y_s^2| \, (1 + |Z_{s-}(\alpha)|^2) \, d\alpha \, ds \right) \\ & \leqslant K_n \int_0^t E(|Y_s^1 - Y_s^2|) \, ds.\end{aligned}$$

Then for each n, one proves in a standard way that there exists a unique solution Y^n to $(SDE)_Z^n$. One remarks moreover that there exists a constant K such that for each n, $\int_0^M |\bar{\varphi}_n(v, w, z)|\, dz \leqslant K(|v|+|w|)$, from which one deduces, using the fact that $E_\alpha(\sup_{t\leqslant T} |Z_t|^2) < \infty$, that

$$\sup_n E(\sup_{t\leqslant T} |Y_t^n|) < \infty.$$

Let us now define the stopping time $\tau_n = \inf\{t > 0, |Y_t^n| \geqslant n\} \wedge T$. It is clear that the sequence τ_n converges to T almost surely, as n tends to infinity. By the uniqueness argument for the solution of $(SDE)_Z^n$, one obtains

$$Y^n_{\tau_n\wedge t} = Y^{n+1}_{\tau_n\wedge t},$$

which allows us to define the process Y in such a way that $Y_{\tau_n\wedge t} = Y^n_{\tau_n\wedge t}$ for each n. That gives finally the existence and uniqueness for a solution of $(SDE)_Z$, and as corollary the existence and uniqueness for $(\mathrm{MP})_{\mathscr{L}(Z)}$. ∎

We are now able to prove the last point (3) of Theorem 3.4. Let us consider a solution Q of (MP), obtained in Proposition 3.8. Let us consider on the probability space $[0, 1]$ a α-process $(W(\alpha))$ with law Q. Let now V be the solution of $(SDE)_W$. Then $\mathscr{L}(V)$ is solution of $(\mathrm{MP})_Q$. But we just have showed in the previous proposition that this martingale problem has a unique solution. Since Q is already a solution of $(\mathrm{MP})_Q$, we deduce that $\mathscr{L}(V) = Q$, which allows us to conclude that (V, W, N, V_0) is a solution of (SDE).

4. A STOCHASTIC PARTICLE APPROXIMATION

In this part, we will introduce some stochastic particle systems and will prove a pathwise propagation of chaos, which will imply the convergence of the empirical measures of the systems to a solution of (2.6). This will be the theorical foundation of the Monte-Carlo algorithm given in the next section.

To define a particle system, we first need to "cutoff" the cross-section, for any particle to have a finite number of collisions before T. Namely we consider

$$B_l(z, \theta) = \psi(z)\, \beta_l(\theta)$$

where

$$\beta_l(\theta) = \beta(\theta)\, \mathbf{1}_{\{|\theta| \geqslant \frac{1}{l}\}}, \tag{4.1}$$

β and ψ satisfying the Hypothesis (R). For the moment, the real number $l > 0$ is fixed, and we set $\|\beta_l\|_1 = \int_{-\pi}^{\pi} \beta_l(\theta)\, d\theta$.

The natural interpretation of the nonlinearity in (2.6) leads to a simple mean field interacting system but a physical interpretation of the equation leads also naturally to a binary mean field interacting particle system. In both cases, these n-particle systems are pure-jump Markov processes with values in $(\mathbb{R}^2)^n$ and with generators defined for $\phi \in C_b((\mathbb{R}^2)^n)$ by

$$\frac{1}{n} \sum_{1 \leqslant i, j \leqslant n} \int_{-\pi}^{\pi} \int_0^M (\phi(v^n + \mathbf{e}_\mathbf{i}.A(\theta)(v_i - v_j)\, \mathbf{1}_{\{z \leqslant \psi(|v_i - v_j|)\}}) - \phi(v^n))\, dz\, \beta_l(\theta)\, d\theta \tag{4.2}$$

for the simple mean-field interacting system and by

$$\begin{aligned} \frac{1}{n} \sum_{1 \leqslant i, j \leqslant n} \int_{-\pi}^{\pi} \int_0^M \frac{1}{2} (\phi(v^n + \mathbf{e}_\mathbf{i}.A(\theta)(v_i - v_j)\, \mathbf{1}_{\{z \leqslant \psi(|v_i - v_j|)\}} \\ + \mathbf{e}_\mathbf{j}.A(\theta)(v_j - v_i)\, \mathbf{1}_{\{z \leqslant \psi(|v_i - v_j|)\}}) - \phi(v^n))\, dz\, \beta_l(\theta)\, d\theta \end{aligned} \tag{4.3}$$

for the binary mean-field interacting system. In these formulas, $v^n = (v_1, ..., v_n)$ denotes the generic point of $(\mathbb{R}^2)^n$ and $\mathbf{e}_\mathbf{i}: h \in \mathbb{R}^2 \mapsto \mathbf{e}_\mathbf{i}.h = (0, ..., 0, h, 0, ..., 0) \in (\mathbb{R}^2)^n$ with h at the ith place.

Both cases can be treated indifferently in a probabilistic point of view. The first particle system can be related to the Nanbu algorithm (cf. ref. 15) and is as simple as possible. The second one can be related to the Bird algorithm (cf. ref. 20). Its main interest is that it conserves momentum and kinetic energy. Moreover a set of numerical experiments shows it looks faster and more precise. We thus consider from now on the **binary mean-field system**. We denote by

$$V^{l,n} = (V^{l,1n}, ..., V^{l,nn})$$

the Markov process defined by (4.3).

We consider as in the previous section a pathwise representation of such processes using Poisson point measures. More precisely, we introduce a family of independent Poisson point measures $(N^{l,ij})_{1 \leqslant i < j \leqslant n}$ on $[0, T] \times [0, M] \times [-\pi, \pi]$ with intensities $\frac{1}{2(n-1)} \beta^l(\theta)\, d\theta\, dz\, dt$. For $i > j$, we set $N^{l,ij} = N^{l,ji}$. Now we consider the process $(V^{l,in})_{1 \leqslant i \leqslant n}$ solution of the following stochastic differential equation:

$$\begin{aligned} V_t^{l,in} = V_0^i + \sum_{j=1}^n \int_0^t \int_0^M \int_{-\pi}^{\pi} A(\theta)(V_{s-}^{l,in} - V_{s-}^{l,jn}) \\ \times \mathbf{1}_{\{z \leqslant \psi(|V_{s-}^{l,in} - V_{s-}^{l,jn}|)\}} N^{l,ij}(d\theta, dz, ds). \end{aligned} \tag{4.4}$$

We construct it easily by working recursively on each interjump interval of the point process $(N^{l,ij})_{1\leqslant i,j\leqslant n}$. It is a n-dimensional Markov process with generator the one described above. Let us denote

$$\mu^{l,n}=\frac{1}{n}\sum_{i=1}^{n}\delta_{V^{l,in}}$$

the empirical measure of this system and by $(\pi^{n,l})_n$ the sequence of laws of $\mu^{l,n}$, which are probability measures on $\mathscr{P}(\mathbb{D}([0,T],\mathbb{R}^2))$.

Theorem 4.1. Assume (R) and $Q_0\in\mathscr{P}_2(\mathbb{R}^2)$. Let $(V_0^i)_{i\geqslant 1}$ be i.i.d. Q_0-distributed random variables. Then the sequence $(\pi^{n,l})_{l,n}$ is uniformly tight for the weak convergence and any limit point charges only probability measures which are solutions of (MP). Thus any limit point (for the convergence in law) of the sequence $(\mu^{l,n})$ is a solution of (MP).

Proof. To prove this theorem, we will show

(1) the tightness of $(\pi^{n,l})_n$ in $\mathscr{P}(\mathscr{P}(\mathbb{D}([0,T],\mathbb{R}^2)))$,

(2) the identification of the limiting values of $(\pi^{n,l})_{l,n}$ as solutions of the nonlinear martingale problem (MP).

One knows (cf. ref. 14, Lemma 4.5) that the tightness of $(\pi^{n,l})_{l,n}$ is equivalent to the tightness of the laws of the semimartingales $V^{l,1n}$ belonging to $\mathscr{P}(\mathbb{D}([0,T],\mathbb{R}^2))$. This tightness can be proved by showing the tightness of the law of the supremum of $|V_t^{l,1n}|$ on $[0,T]$ and the the Aldous criterion for $V^{l,1n}$.

One easily proves by a good use of Burkholder–Davis–Gundy and Doob's inequalities for (4.4) and thanks to (R) that

$$\sup_{l,n}E(\sup_{t\leqslant T}|V_t^{l,1n}|^2)<+\infty. \tag{4.5}$$

from which we deduce without difficulty the tightness of the laws of $V^{l,1n}$ and hence the tightness of the sequence $(\pi^{l,n})$.

Let us now prove that all the limit values are solutions of the nonlinear martingale problem (MP). Consider $\pi^\infty\in\mathscr{P}(\mathscr{P}(\mathbb{D}([0,T],\mathbb{R}^2)))$ a limit value of $(\pi^{l,n})$. It is the limit point of a subsequence we still denote by $(\pi^{l,n})$.

For $\phi\in C_b^1(\mathbb{R}^2)$, $0\leqslant s_1,\dots,s_k\leqslant s<t$, $g_1,\dots,g_k\in C_b(\mathbb{R}^2)$, $Q\in\mathscr{P}(\mathbb{D}([0,T],\mathbb{R}^2))$ and for X the canonical process on $\mathbb{D}([0,T],\mathbb{R}^2)$, we set

$$F(Q)=\left\langle g_1(X_{s_1})\cdots g_k(X_{s_k})\left(\phi(X_t)-\phi(X_s)-\int_s^t\int_0^M\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\right.\right.$$
$$\left.\left.\times\Big(\phi(X_u+A(\theta)\,\varphi(X_u,w,z))-\phi(X_u)\Big)\,Q_u(dw)\,\beta(\theta)\,d\theta dz\,du\right),Q\right\rangle$$
$$=\langle g_1(X_{s_1})\cdots g_k(X_{s_k})(H_t^{\phi}-H_s^{\phi}),Q\rangle \tag{4.6}$$

with the notation used in the proof of Proposition (3.8).

Our aim is to prove that $\langle |F|,\pi^{\infty}\rangle=0$. The mapping F is not continuous since the projections $X\mapsto X_t$ are not continuous for the Skorohod topology. However, for any $Q\in\mathscr{P}(\mathbb{D}([0,T],\mathbb{R}^2))$, $X\mapsto X_t$ is Q-almost surely continuous for all t outside an at most countable set D_Q, and then F is continuous at the point Q if $s, t, s_1,\ldots, s_k$ are not in D_Q. Here we use the continuity and the boundedness of $\phi, g_1,\ldots, g_k$ and also the continuity of $(q,v)\mapsto\int_0^M\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}(\phi(v+A(\theta)\,\varphi(v,w,z))-\phi(v))q(dw)\,\beta(\theta)\,d\theta\,dz$ on $\mathscr{P}(\mathbb{D}([0,T],\mathbb{R}^2))\times\mathbb{R}^2$. Now one can show that the set D of all t for which $\pi^{\infty}(Q, t\in D_Q)>0$ is again at most countable. Thus, if $s, t, s_1,\ldots, s_k$ are in D^c, F is π^{∞}-a.s. continuous. Then,

$$\langle F^2,\pi^{\infty}\rangle=\lim_{l,n}\langle F^2,\pi^{l,n}\rangle$$

But $\langle |F|,\pi^{l,n}\rangle\leqslant\langle |F^l|,\pi^{l,n}\rangle+\langle |F-F^l|,\pi^{l,n}\rangle$ where F^l is defined as F with β^l instead of β. Firstly,

$$\begin{aligned}\langle (F^l)^2,\pi^{l,n}\rangle&=E((F^l(\mu^{l,n}))^2)\\&=E\left(\left(\frac{1}{n}\sum_{i=1}^{n}(M_t^{l,i\phi}-M_s^{l,i\phi})\,g_1(V_{s_1}^{l,in})\cdots g_k(V_{s_k}^{l,in})\right)^2\right)\\&=\frac{1}{n}E(((M_t^{l,1\phi}-M_s^{l,1\phi})\,g_1(V_{s_1}^{l,1n})\cdots g_k(V_{s_k}^{l,1n}))^2)\\&\quad+\frac{n-1}{n}E((M_t^{l,1\phi}-M_s^{l,1\phi})(M_t^{l,2\phi}-M_s^{l,2\phi})\\&\quad\times g_1(V_{s_1}^{l,1n})\cdots g_k(V_{s_k}^{l,1n})\,g_1(V_{s_1}^{l,2n})\cdots g_k(V_{s_k}^{l,2n}))\end{aligned}\tag{4.7}$$

where $M^{l,i\phi}$ is the martingale defined by

$$M_t^{l,i\phi}=\phi(V_t^{l,in})-\phi(V_0^{i})-\frac{1}{n}\sum_{j=1}^{n}\int_0^t\int_0^M\int_{-\pi}^{\pi}$$
$$\times(\phi(V_s^{l,in}+A(\theta)\,\varphi(V_s^{l,in},V_s^{l,jn},z))-\phi(V_s^{l,in}))\,\beta^l(\theta)\,d\theta\,dz\,ds$$

and with Doob–Meyer process given by

$$\langle M^{l,i\phi}\rangle_t = \frac{1}{n}\sum_{j=1}^{n}\int_0^t\int_0^M\int_{-\pi}^{\pi} \times (\phi(V_s^{l,in}+A(\theta)\,\varphi(V_s^{l,in},V_s^{l,jn},z))-\phi(V_s^{l,in}))^2\,\beta^l(\theta)\,d\theta\,dz\,ds$$

and for $i\neq j$,

$$\langle M^{l,i\phi}, M^{l,j\phi}\rangle_t = \frac{1}{n}\int_0^t\int_0^M\int_{-\pi}^{\pi}(\phi(V_s^{l,in}+A(\theta)\,\varphi(V_s^{l,in},V_s^{l,jn},z))-\phi(V_s^{l,in})) \times (\phi(V_s^{l,jn}+A(\theta)\,\varphi(V_s^{l,jn},V_s^{l,in},z))-\phi(V_s^{l,jn}))\,\beta^l(\theta)\,d\theta\,dz\,ds.$$

The right terms in (4.7) go to 0 thanks to the expression of the Doob–Meyer process, to the uniform integrability proved in (4.5) and thanks to hypothesis (R). Moreover the convergence is uniform in l. Hence

$$\lim_n \langle |F^l|, \pi^{l,n}\rangle = 0, \text{ uniformly in } l.$$

Otherwise,

$$\begin{aligned}\langle |F-F^l|, \pi^{l,n}\rangle &= E(|F-F^l|\,(\mu^{l,n}))\\ &= E\left(\left|\left\langle \int_s^t\int_0^M\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}(\phi(X_u+A(\theta)\,\varphi(X_u,w,z))-\phi(X_u))\right.\right.\right.\\ &\qquad\left.\left.\left.\times\,\mu_u^{l,n}(dw)(\beta(\theta)-\beta^l(\theta))\,d\theta\,dz\,du,\ \mu^{l,n}\right\rangle\right|\right)\\ &\leqslant K_l\sup_{l,n} E(\sup_{t\leqslant T}\langle |v|,\mu_t^{l,n}\rangle)\leqslant K_l\sup_{l,n}(E(\sup_{t\leqslant T}\langle |v|^2,\mu_t^{l,n}\rangle))^{\frac{1}{2}}\end{aligned}$$

The second term is finite by (4.5) and $K_l = C^{te}\int_{-\pi}^{\pi}|\theta|\,|\beta(\theta)-\beta^l(\theta)|\,d\theta$ tends to 0 as l tends to infinity.

We have then proved that

$$\langle |F|, \pi^\infty\rangle = 0.$$

Thus, $F(Q)$ is π^∞-a.s. equal to 0, for every $s, t, s_1,\dots, s_k$ outside of the countable set D_Q. It is sufficient to assure that π^∞-a.s., Q is solution of the nonlinear martingale problem (MP). ∎

Corollary 4.2. Assume (R) and $Q_0 \in \mathscr{P}_2(\mathbb{R}^2)$ and consider a sequence μ^{l_r, n_r} which converges to Q. Then the probability measure-valued process $(\mu_t^{l_r, n_r})_{t \geqslant 0}$ converges in probability to the flow $(Q_t)_{t \geqslant 0}$ in the space $\mathbb{D}([0, T], \mathscr{P}(\mathbb{R}^2))$ endowed with the uniform topology.

Proof. The flow $(Q_t)_{t \geqslant 0}$ is deterministic and continuous. Then the convergence to $(Q_t)_{t \geqslant 0}$ is the same for the Skorohod or for the uniform topology. We use an intermediary lemma, proved in Méléard,[14] Lemma 4.8 (see also Léonard[13]).

Lemma 4.3. Let $(\mu^n)_n$ be a sequence of random probability measures on $\mathbb{D}_T$ which converges in law to a deterministic probability measure Q in $\mathscr{P}_2(\mathbb{D}_T)$. Let us assume moreover that

$$\lim_{r \to 0} \sup_{0 \leqslant t \leqslant T} E_Q\Big(\sup_{t-r<s<t+r} |\Delta X_s| \wedge 1\Big) = 0 \tag{4.8}$$

where X is the canonical process on $\mathbb{D}_T$, then the flow $(\mu_t^n)_{t \geqslant 0}$ converges in probability to $(Q_t)_{t \geqslant 0}$ in $\mathbb{D}([0, T], \mathscr{P}(\mathbb{R}^2))$ endowed with the uniform topology.

This result is not obvious since in $\mathbb{D}_T$ the projections are not continuous for the Skorohod topology.

Let us verify (4.8) in our context. We know by the point (ii) of Theorem 3.3 that X can be obtained on an enlarged probability space as solution of (3.1). Then

$$\begin{aligned}
E_Q\Big(&\sup_{t-r<s<t+r} |\Delta X_s| \wedge 1\Big) \\
&\leqslant E_Q\left(\sum_{s \in [t-r, t+r]} |\Delta X_s| \wedge 1 \right) \\
&\leqslant \int_{t-r}^{t+r} \int_0^1 \int_0^M \int_{-\pi}^{\pi} |A(\theta)|\, |\varphi(X_{s-}, W_{s-}(\alpha), z)|\, \beta(\theta)\, d\theta\, dz\, d\alpha\, ds \\
&\leqslant K \int_{t-r}^{t+r} \int_0^1 E_Q((|X_{s-} - W_{s-}(\alpha)|)\, d\alpha\, ds \\
&\leqslant K r E_Q(\sup_{t \leqslant t} |X_t|).
\end{aligned}$$

But this last quantity tends to 0 as r tends to 0 since $E_Q(\sup_{t \leqslant T} |X_t|)$ is finite. Indeed, since $Q \in \mathscr{P}_2(\mathbb{D}_T)$, the canonical process X is a L_T^2-process under Q. We have the result. ∎

We will now explicit the algorithm of simulation.

5. THE MONTE-CARLO ALGORITHM

We deduce from the above study an algorithm associated with the binary mean-field interacting particle system (Bird's approach). We could do the same thing with the simple mean-field interacting particle system (Nanbu's approach), but the numerical results seem less efficient.

From now on, the functions ψ and β defining the cross-section B, the initial distribution Q_0, the terminal time $T>0$, the size $n \geqslant 2$ of the particle system and the cutoff parameter $l>0$ are fixed. We denote by $B_l(z,\theta)=\psi(z)\,\beta_l(\theta)$ the corresponding cross-section with cutoff. Because of Theorem 4.1, we simulate a particle system following (4.3), i.e., the whole path $(V^n_t)_{t\in[0,T]} \in \mathbb{D}([0,T],(\mathbb{R}^2)^n)$.

First of all, we assume that V^n_0 is simulated according to the initial distribution $Q_0^{\otimes n}$. Then, we denote by $0<T_1<\cdots<T_k$ the successive jump times until T of a standard Poisson process with parameter $\frac{nM\,\|\beta_l\|_1}{2}$. For example, one simulates independent exponential laws with this rate which describe the inter-collision time-intervals.

Before the first collision, the velocities do not change, so that we set $V^n_s=V^n_0$ for all $s<T_1$. Let us describe the first collision. We choose at random a couple (i,j) of particles according a uniform law over $\{(l,m)\in\{1,\ldots,n\}^2;\,m\neq l\}$,. We choose z uniformly on the interval $[0,M]$, and we finally choose the collision angle following the law $\frac{\beta_l(\theta)}{\|\beta_l\|_1}\,d\theta$. Then we set

$$\begin{aligned}
V^{n,i}_{T_1} &= V^{n,i}_0 + A(\theta)(V^{n,i}_0 - V^{n,j}_0)\,\mathbf{1}_{\{z\leqslant\psi(|V^{n,i}_0-V^{n,j}_0|)\}}\\
V^{n,j}_{T_1} &= V^{n,j}_0 + A(\theta)(V^{n,j}_0 - V^{n,i}_0)\,\mathbf{1}_{\{z\leqslant\psi(|V^{n,i}_0-V^{n,j}_0|)\}}\\
V^{n,l}_{T_1} &= V^{n,l}_0 \qquad \text{if} \quad l\neq\{i,j\}
\end{aligned}$$

Since nothing happens between T_1 and T_2, we set $V^n_s=V^n_{T_1}$ for all $s\in[T_1,T_2[$.

Iterating this method, we simulate $V^n_{T_1}, V^n_{T_2},\ldots, V^n_{T_k}$, i.e., the whole path $(V^n_t)_{t\in[0,T]}$, which was our aim.

Notice that this algorithm is very simple and takes a few lines of program and does not require to discretize time. It furthermore conserves momentum and kinetic energy.

6. NUMERICAL STUDY

We now would like to have an idea about the true speeds of convergence of the previous algorithm in physical situations. In such a situation, the function ψ does not satisfy assumption (R), since it is not bounded. We thus will have to replace ψ by an approximating bounded function ψ_M. Another problem occurs: we do not know if the uniqueness holds for our Boltzmann equation.

Of course, numerical results will not allow to conclude anything, but we will see that in the following study, neither the uniqueness nor the boundeness of ψ will be a problem.

Let us now be precise. We consider the following initial distribution of the velocities:

$$P_0(dv) = 1_{[-1/2,\,1/2]^2}(v)\,dv$$

and the physical cross section corresponding to interactions in $1/r^4$:

$$B(z, \theta) = \psi(z)\,\beta(\theta) = z^{-1/3}\theta^{-5/3}$$

For $M > 0$ and $l > 0$, we set

$$\psi_M(z) = \psi(z) \wedge M \qquad \text{and} \qquad \beta_l(\theta) = \beta(\theta)\,1_{|\theta| \geqslant 1/l}$$

Notice that $B^M = \psi_M \beta$ satisfies assumption (R) and that $B^{M,l} = \psi_M \beta_l$ is its corresponding cross section with cutoff.

For each M, l, we denote by $\{Q^{M,l}\}$ the solution of the martingale problem with the cross section $B^{M,l}$, obtained by Theorem 4.1. We know that for each M, each l, $\{Q^{M,l}\}$ is the limit, as n tends to infinity, of the empirical measures $\mu^{M,l,n}$ associated with the simulable empirical particle systems. We also know that for each fixed M, $\{Q^{M,l}\}_l$ is tight, and that any limit point Q^M is solution of the martingale problem with the cross section B^M.

In order to study these many convergences, we have to consider a fixed quantity. The first idea is to study the moments of the solution of the Boltzmann equation. But in some situations, we are able to prove the uniqueness of the moments of the solutions of the Boltzmann equation, although the uniqueness of solutions stays an open problem. We thus consider the following quantities, for $t_0 = 1$:

$$m^M(t_0) = \int_{\mathbb{R}^2} |v|^4\, e^{-|v|^2} Q^M_{t_0}(dv); \qquad m^{M,l}(t_0) = \int_{\mathbb{R}^2} |v|^4\, e^{-|v|^2} Q^{M,l}_{t_0}(dv)$$

$$\text{and} \quad m^{M,l,n}(t_0) = \int_{\mathbb{R}^2} |v|^4\, e^{-|v|^2} \mu^{M,l,n}_{t_0}(dv)$$

First of all, we study the possible convergence of $m^{M,l}(t_0)$, as M and l go to infinity. We compute numerically this quantity, by using Corollary 4.2: we

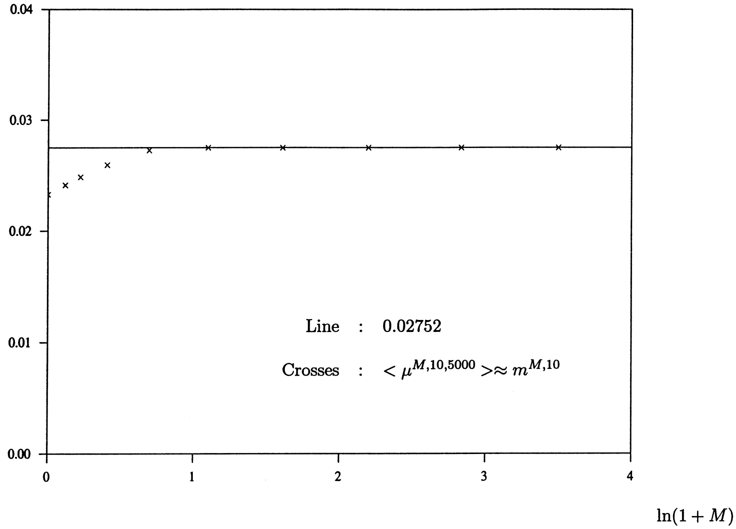

Fig. 1. "True value" of $m^{M,10}$ as a function of $\ln(1+M)$.

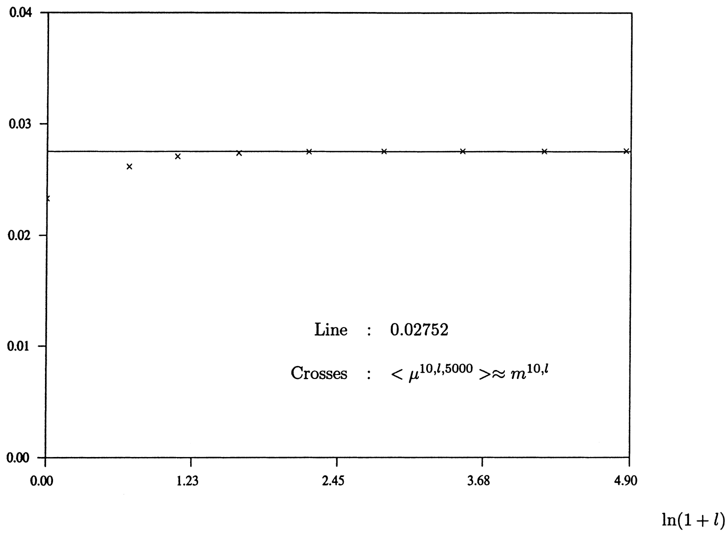

Fig. 2. "True value" of $m^{10,l}$ as a function of $\ln(1+l)$.

are allowed to say that $m^{M,l} \approx \langle \mu^{M,l,5000} \rangle$, where $\langle \cdot \rangle$ denotes an "empirical" mean over many experiences.

We obtain Figs. 1 and 2, which seem to show that $m^{M,l}$ converges, very fastly, to some quantity, which we denote by $m(t_0)$, which equals 0, 02752, and which we hope to be equal to $\int_{\mathbb{R}^2} |v|^4 e^{-|v|^2} Q_{t_0}(dv)$, where Q is a (possibly unique) solution of the martingale problem with the physical cross section B.

Let us mention that in the Maxwellian case, i.e., when $\psi \equiv 1$, we are able to prove that the rate of convergence of Q^l to Q is (at the worse) proportional to $\int_0^{1/l} \theta^2 \beta(\theta)\, d\theta$. Numerical experiments (see ref. 6) confirm this speed of convergence. Here, we thus would expect a speed of convergence of $m^{M,l}(t_0)$ to $m^M(t_0)$ in $\int_0^{1/l} \theta^2 \beta(\theta)\, d\theta \approx 1/l^{4/3}$, at least for M fixed. We obtain the following values of the error $e(l, M)$, in percent, of $m^{M,l}(t_0)$ with respect to $m^M(t_0)$:

l	1	2	4	8
$e(l, 10)$	4.94	1.63	0.51	0.08
$e(l, 10) \times l^{4/3}$	4.94	4.10	3.23	1.28

(We do not consider large values of l, because for l large, the error due to the computations become larger than that due to the cutoff). We thus see that the speed of convergence might hold (here it seems to be faster, but this must be related to our choice of the functional $|v|^4 e^{-|v|^2}$).

We now would like to study the "mean" speed of convergence of $m^{M,l,n}(t_0)$ to $m^{M,l}(t_0)$, as n goes to infinity. We thus denote by $e^{M,l,n}$ the mean error, in percent, of $m^{M,l,n}(t_0)$ (obtained by one simulation), with respect to $m^{M,l}(t_0)$. In other words, for $\langle \cdot \rangle$ the mean over several experiences,

$$e^{M,l,n}(t_0) = \left\langle 100 \times \left| \frac{m^{M,l,n}(t_0) - m^{M,l}(t_0)}{m^{M,l}(t_0)} \right| \right\rangle$$

Considering that $m^{10,10}(t_0) = m^{100,100}(t_0) = m(t_0) = 0.02752$, we obtain Fig. 3.

It thus seems that the mean error is in $K/\sqrt{n}$, with the constant K not depending too much on M and l, at least for M and l sufficiently large.

We have proved, in the Maxwellian case, see ref. 7, a fluctuation Theorem for each l fixed, which still seems to hold here, see Fig. 3. But

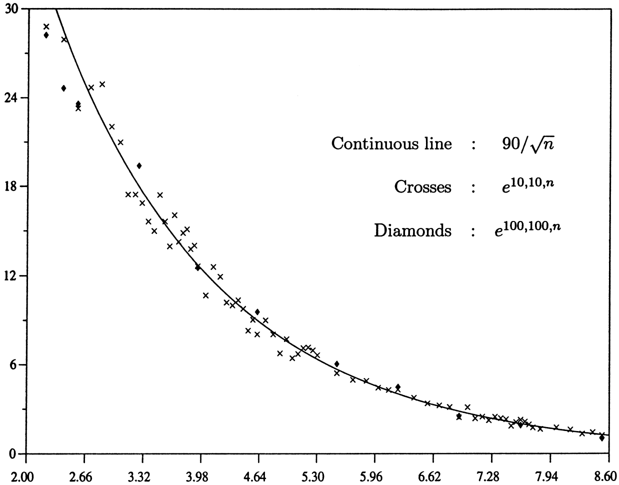

Fig. 3. Mean error of one simulation as a function of $\ln(1+n)$.

once again, the fact that the constant $K/\sqrt{n}$ does not depend on M and l seems to be related to our choice for the quantity $m(t_0)$.

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